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1983 J. Phys. A: Math. Gen. 16 3485

(http://iopscience.iop.org/0305-4470/16/15/014)

J. Phys. A: Math. Gen. 16 (1983) 3485–3492. Printed in Great Britain

Non-associative properties of a deformed Poisson product and the quantisation problem

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Received 8 February 1983

Abstract. We discuss the associativity properties of a deformed algebra of phase–space functions, the deformation taken to finite order. We assess our findings in view of a recently proposed quantisation scheme which admits non-associative algebras of operators. We draw a number of conclusions pertaining to the applicability of the deformed algebra for quantum mechanical descriptions.

1. Introduction

In a recent paper (Ktorides and Papaloucas 1982) we have investigated the derivation properties of a certain deformed Poisson algebra and related our findings to Joseph's quantisation prescription (Joseph 1970). The deformed Poisson product has been introduced, for quantisation purposes, by Bayen *et al* (1978) and in its most general form reads

$$\{f, g\}_\lambda = \sum_{k=0}^{\infty} \lambda^k C_k(f, g) \tag{1.1}$$

where f, g are complex-valued polynomial functions on phase space and the C_k are certain two-cochains (Gote and Grosshans 1978) in the space of polynomial functions. We have been particularly interested in the infinitesimal form of (1.1) which pertains directly to the Poisson product $\{f, g\}$. It reads

$$\{f, g\} \to \{f, g\}_\lambda = \{f, g\} + \lambda C_1(f, g). \tag{1.2}$$

Our result was that Joseph's quantisation requirement can be met, for product (1.2), if the two-cochain $C_1(f, g)$ obeys a certain condition (Ktorides and Papaloucas 1982).

In the present paper we shall investigate a specific realisation of the deformation product (1.1) which has been proposed by Bayen *et al* (1978). We shall be particularly interested in its associative properties.

In § 2 we investigate these properties for the case where the deformation is of first order. We relate our findings to the recently proposed quantisation scheme by Okubo (1981). In § 3 we study deformations of finite order greater than one and in § 4 we formulate our conclusions.

2. A specific realisation of the deformation product and related associativity considerations

A particular realisation of (1.2) can be achieved by deforming the customary (associative) multiplication between f and g, as follows:

$$fg \rightarrow (f \cdot g)_\lambda = fg + \lambda\{f, g\}. \tag{2.1}$$

With respect to the above product, let us introduce the associator (f, g, h)

$$(f, g, h)_\lambda = ((f \cdot g)_\lambda \cdot h)_\lambda - (f \cdot (g \cdot h)_\lambda)_\lambda. \tag{2.2}$$

It can be seen, from this definition, that product (2.1) is associative only if its associator vanishes. One finds

$$(f, g, h)_\lambda = \lambda^2\{\{f, h\}, g\}. \tag{2.3}$$

Given that associativity is a revered property in quantum mechainical formulations the above result calls for closer inspection.

One way to proceed at this point is to note that the associator is of second order in λ, while the product deformation, as given by (2.1), is of first order. Referring to (1.1), one could consider the possibility of extending the deformation prescription given by (2.1) to higher orders in λ, say λ^k, in such a way that the order of the non-vanishing associator be pushed up further (to, at least, order λ^{k+1}). It is conceivable that in the limit $k \rightarrow \infty$, a suitably constructed deformation will have a vanishing associator. Bayen *et al* (1978) have been able to construct such a product, the so-called star product. It reads

$$f*g = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} P^r(f, g) = \exp(\lambda P)(f, g). \tag{2.4}$$

In the above expression $P^r(f, g)$ denotes the bidifferential operator

$$P^r(f, g) = \Omega^{i_1 j_1} \cdots \Omega^{i_r j_r} \nabla_{i_1 \ldots i_r} f \nabla_{j_1 \ldots j_r} g \tag{2.5}$$

which can be viewed as an rth-order Poisson differential operator. For example, for the familiar flat phase space of classical mechanics one has

$$P(f, g) = (\partial f/\partial s^i)\omega_{ij}\partial g/\partial s^j,$$

$$P^2(f, g) = \frac{\partial^2 f}{\partial s^{i_1}\partial^{i_2}} \omega_{i_1 j_1}\omega_{i_2 j_2} \frac{\partial^2 g}{\partial s^{j_1}\partial s^{j_2}}, \qquad \text{etc} \tag{2.6}$$

where $s^i = (q^1, \ldots, q^n, p^1, \ldots, p^n)$ are the phase–space coordinates and ω_{ij} is the familiar symplectic metric

$$\omega_{ij} = \begin{pmatrix} 0 & -I_{nxn} \\ I_{nxn} & 0 \end{pmatrix}. \tag{2.7}$$

Note that, to order λ^1, the star product (2.4) reduces to the deformation (2.1). Finally it should be pointed out that (2.4) becomes the Moyal product (Moyal 1949) for $\lambda = \frac{1}{2}i\hbar$.

The restoration of associativity is certainly welcome. On the other hand, product (2.4) could be criticised on the following grounds. The majority of quantum mechanical calculations are done via approximation methods. For example, in semiclassical and/or WKB approximation methods, calculations are performed to finite order in $\hbar$.

Only systems with integrable classical counterparts (such as the harmonic oscillator, the Kepler two-body problem, the sine-Gordon model etc) posses a complete quantum mechanical solution. But finite order in $\hbar$ is, at least from the viewpoint of the Moyal product, equivalent to saying finite order in λ which, in turn, inevitably brings non-associativity back into the picture. Even if the non-associative terms are pushed to the next order in λ, the fact remains that non-associativity is lurking in the background and its cumulative effects may not vanish. Moreover, when the phase-space manifold has curvature and/or torsion the non-associative effects could very well be of the same order in λ as the deformation itself (beginning from order λ^2 if there is torsion and order λ^3 if there is torsion and curvature) (Bayen *et al* 1978).

The above arguments bring us to a second alternative, namely to allow for the possibility of finite-order calculations and face non-associativity directly. The plausibility for employing non-associative quantum mechanical algebras has been raised in a recent paper by Okubo (1981). This author has distinguished between the wider algebra $\mathscr{A}$ of quantum mechanical operators and the algebra $\mathscr{A}_0$ of operators corresponding to observables (Hermitian operators). This distinction arises naturally as follows. Let $A, B \in \mathscr{A}_0$. It does not follow that $AB \in \mathscr{A}_0$, only that $\frac{1}{2}(AB+BA) \in \mathscr{A}_0$. So the product AB should, in general, be accommodated in the larger algebra $\mathscr{A}$ of operators. Okubo proceeded to investigate what must be the minimal properties of $\mathscr{A}$ such that the Heisenberg equation of motion

$$\mathrm{d}A/\mathrm{d}t = (\mathrm{i}/\hbar)[H, A] \tag{2.8}$$

holds its validity for any observable $A \in \mathscr{A}_0$. This equation implies, in particular, that $\mathrm{ad}(H)$ acts as a derivation on $\mathscr{A}$:

$$[H, AB] = [H, A]B + A[H, B]. \tag{2.9}$$

With such a minimal restriction it turns out that $\mathscr{A}$ need not be associative. According to a theorem proved by Myung (1972) the only requirements placed by (2.9) on $\mathscr{A}$ are that

(i) $\mathscr{A}$ be flexible, which means

$$(A, B, C) = -(C, B, A) \tag{2.10}$$

where (A, B, C) denotes the associator in $\mathscr{A}$, and

(ii) $\mathscr{A}$ be Lie admissible, which means

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0. \tag{2.11}$$

Clearly both properties are automatically guaranteed if the product AB, in $\mathscr{A}$, is associative. However, in general, $\mathscr{A}$ need not be associative. As long as relations (2.10) and (2.11) hold true for its product, $\mathscr{A}$ is a legitimate candidate for a quantum mechanical operator algebra. Indeed, Okubo has been able to formulate a strong quantisation scheme which maps classical phase-space functions into a non-associative (but flexible and Lie-admissible) algebra of operators.

Armed with the above information, let us now test whether the non-associative deformed product (2.1) fulfils Okubo's requirements. From (2.3) it follows that

$$(h, g, f)_\lambda = \lambda^2\{\{h, f\}, g\} - \lambda^2\{\{f, h\}, g\} \tag{2.12}$$

which verifies flexibility.

Lie admissibility can also be checked in a straight forward manner. One has to calculate

$$[[f, g]_\lambda, h]_\lambda = ((f \cdot g)_\lambda \cdot h)_\lambda - (h \cdot (f \cdot g)_\lambda)_\lambda - ((g \cdot f)_\lambda \cdot h)_\lambda + (h \cdot (g \cdot f)_\lambda)_\lambda.$$

One finds

$$[[f, g]_\lambda, h]_\lambda = 4\lambda^2\{\{f, g\}, h\} \tag{2.13}$$

i.e. the double commutator of the deformed product reduces to a familiar double Poisson product. The Jacobi identity is now an immediate consequence of (2.13). Explicitly,

$$[[f, g]_\lambda, h]_\lambda + [[g, h]_\lambda, f]_\lambda + [[h, f]_\lambda, g]_\lambda = 4\lambda^2(\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}) = 0 \tag{2.14}$$

which verifies the Lie admissibility of the deformed product (2.1). This fact has also been noted by Okubo in relation to the Kokoris nodal algebra (see Schafer 1966).

3. Arbitrary finite-order deformations

Encouraged by the first-order result obtained in § 2 we shall now proceed to investigate the case when the product deformation is taken to some arbitrary finite-order k. To be explicit, we shall consider the product

$$(f, g)_{\lambda^k} = \sum_{r=0}^{k} \frac{\lambda^r}{r!} P^r(f, g), \qquad k < \infty. \tag{3.1}$$

The associator $(f, g, h)_{\lambda^k}$ is not zero, as one might have expected. Indeed, it is given by

$$(f, g, h)_{\lambda^k} = \sum_{s=0}^{k} \sum_{r=0}^{k} \frac{\lambda^{r+s}}{s!r!} [P^s(P^r(f, g), h) - P^s(f, P^r(g, h))] \tag{3.2}$$

and cannot possibly vanish identically if only because of the λ^{2k} term. The latter reads

$$\frac{\lambda^{2k}}{(k!)^2} [P^k(P^k(f, g), h) - P^k(f, P^k(g, h))].$$

Now in the expansion of $P^k(P^k(f, g), h)$ terms will occur where in a $2k$th-order derivative of f (in p's and q's) multiplies a kth-order derivative of g times a kth-order derivative of h; such terms are not contained in the expansion of $P^k(f, P^k(g, h))$ wherein only kth-order derivatives of f enter.

We shall now prove that all terms in the double sum (3.2) up to order λ^k (i.e. $s + r \leq k$) vanish, implying that non-associativity is of order λ^{k+1} or higher. To this end, we consider a generalised form of (2.1), as follows:

$$(f, g)_{\lambda^k} = \sum_{r=0}^{k} a_r \frac{\lambda^r}{r!} P^r(f, g). \tag{3.3}$$

The study of the above series to infinite order ($k \to \infty$) by Bayen *et al* (1978) yielded the following result. Associativity imposes on the a_r the restrictions

$$a_r a_{t-r} = a_s a_{t-s}, \qquad s + r \leq t, \qquad t = 0, 1, 2, \ldots . \tag{3.4}$$

The above condition can be consistently fulfilled in the case of an infinite-order expansion by setting all $a_r = 1$ (leading precisely to the star product (2.4)). To finite order, on the other hand, condition (3.4) cannot be consistently fulfilled. Indeed, in the finite-order case t runs up to $2k$ so that (3.4) places a restriction on $2k$ coefficients $a_1, a_2, \ldots, a_{2k}$. But only the first k coefficients $a_1, \ldots, a_k$ are available in the original expansion (3.3). The latter can be consistently fixed via (3.4), which means that terms up to order λ^k in the associator can be surely made to vanish (one has to set $a_1 = a_2 = \ldots = a_k = 1$ as we already did in (3.1)). Terms of order λ^{k+1} and higher, on the other hand, are outside the jurisdiction of condition (3.4). Therefore, they do not vanish necessarily. In fact, we have already seen that the λ^{2k} term in the associator certainly does not vanish. We have explicitly checked the case when $k = 2$ (deformation to second order) and have found that the λ^3 and λ^4 terms in (3.2) do not vanish.

According to Okubo, the lack of associativity does not, in itself, constitute a reason to abandon the finite-order deformation (3.1) as a viable quantum mechanical product. Provided that (3.1) is flexible and Lie admissible it can be employed for quantum mechanical calculations.

We shall proceed to examine flexibility and Lie admissibility for the product (3.1) as soon as the following two preliminary remarks are out of the way. The first amounts to a notational convention. By $f^{(r)}$ we shall collectively denote any rth-order derivative of f in q's and p's. This notation allows us to write, for example,

$$P^r(f, g) = f^{(r)} g^{(r)} \tag{3.5}$$

and

$$P^r(P^s(f, g), h) = h^{(r)} f^{(r+s)} g^{(s)} + h^{(r)} f^{(r+s-1)} g^{(s+1)} + \ldots + h^{(r)} f^{(s)} g^{(r+s)}. \tag{3.6}$$

Of course $f^{(r)} g^{(r)}$ in (3.5) is shorthand for a number of terms containing an rth-order derivative of f times an rth-order derivative of g with appropriate numerical coefficients, according to formula (2.5). Clearly, half of the total number of derivatives on f and g will be with respect to q and the other half with respect to p. Similar analysis applies to the terms entering the right-hand side of (3.6).

The second remark pertains to symmetry properties of the P^r operation with respect to an exchange of its arguments. It readily follows that

$$P^r(f, g) = -P^r(g, f) \qquad \text{if } r \text{ is odd}$$

and

$$P^r(f, g) = P^r(g, f) \qquad \text{if } r \text{ is even.}$$

We are now in a position to study flexibility. This property demands that (3.2) should coincide with

$$-(h, g, f)_{\lambda^k} = \sum_{r=0}^{k} \sum_{s=0}^{k} \frac{\lambda^{r+s}}{r!\,s!} [P^s(h, P^r(g, f)) - P^s(P^r(h, g), f)]. \tag{3.7}$$

Let us check the possibility of a term by term equality between the sums entering (3.2) and (3.7). For a term in (3.7) with s even, r even rearrangements can be made freely without any sign changes. Explicitly

$$P^s(h, P^r(g, f)) - P^s(P^r(h, g), f) = P^s(P^r(f, g), h) - P^s(f, P^r(g, h)). \tag{3.8}$$

But the right-hand side of (3.8) matches exactly the corresponding term in (3.2). Similarly, for r odd, s odd the double rearrangement will restore the original signs and the corresponding terms in (3.2) and (3.7) match once more. However for s even, r odd—or vice versa—the corresponding terms have exactly opposite signs:

$$P^s(h, P^r(g, f)) - P^s(P^r(h, g), f) = -P^s(P^r(f, g), h) + P^s(f, P^r(g, h)). \tag{3.9}$$

There are two possible ways that flexibility still holds: either that even–odd terms in r and s are absent in the expansion (3.2) and (3.7) or that cancellations occur between terms with different values of r and s but a constant value of $r+s$.

We shall show that neither of these holds by means of a counterexample. Consider the even–odd terms $s=k$, $r=k-1$ and $s=k-1$, $r=k$ occurring in the expansion (3.2). They read

$$\frac{\lambda^{2k-1}}{k!(k-1)!}[P^k(P^{k-1}(f, g), h) - P^k(f, P^{k-1}(g, h)) + P^{k-1}(P^k(f, g), h) - P^{k-1}(f, P^k(g, h))]. \tag{3.10}$$

Clearly, with the above expression we have exhausted all the λ^{2k-1}-order terms in (3.2). Employing our previously introduced notation, (3.10) reads as

$$\begin{aligned}\frac{\lambda^{2k-1}}{k!(k-1)!}[h^{(k)}&(f^{(k-1)}g^{(2k-1)} + f^{(k)}g^{(2k-2)} + \ldots + f^{(2k-1)}g^{(k-1)})\\ &-f^{(k)}(g^{(k-1)}h^{(2k-1)} + \ldots + g^{(2k-1)}h^{(k-1)})\\ &+h^{(k-1)}(f^{(k)}g^{(2k-1)} + \ldots + f^{(2k-1)}g^{(k)})\\ &-f^{(k-1)}(g^{(k)}h^{(2k-1)} + \ldots + g^{(2k-1)}h^{(k)})].\end{aligned} \tag{3.11}$$

Each row contains $(k+1)$ type of terms. Consider the first row. There are $(k+1)$ different types of terms in each of which h is differentiated to order k. These terms should be cancelled by terms appearing in the other three rows. Upon a glance on the last three rows we find that the type of terms containing $h^{(k)}$ appear only once in the second row (in the form $f^{(k)}g^{(2k-2)}h^{(k)}$), no times in the third row and only once in the fourth row (in the form $f^{(k-1)}g^{(2k-1)}h^{(k)}$). This means that a necessary condition for cancellation of the terms in question is that $k+1=2$, i.e. $k=1$. Therefore flexibility is possible only for the first-order deformation, an occurrence we have explicitly shown in § 2. By the way, the above argument also shows that even–odd type terms do not vanish individually. For this to have been the case in the example just considered, the first two rows would have had to cancel each other. Such an alternative is even more restrictive than the previous one. In particular it would imply $k+1=1$, i.e. the trivial expansion with $k=0$.

Let us now turn to Lie admissibility. To begin with we find

$$[f, g]_{\lambda^k} = (f\cdot g)_{\lambda^k} - (g\cdot f)_{\lambda^k} = \sum_{r=0}^{k}\frac{\lambda^r}{r!}[P^r(f, g) - P^r(g, f)]$$

which means that only the odd terms survive in the expansion. We obtain

$$[f, g]_{\lambda^k} = \sum_{r=0}^{k}\frac{2\lambda^{2r+1}}{(2r+1)!}P^{2r+1}(f, g) \tag{3.12}$$

with $k' = k/2$ if k is even and $k' = k - \frac{1}{2}$ if k is odd. Lie admissibility means

$$[[f, g]_{\lambda^k}, h]_{\lambda^k} + [[g, h]_{\lambda^k}, f]_{\lambda^k} + [[h, f]_{\lambda^k}, g]_{\lambda^k}$$
$$= \sum_{r=0}^{k'} \sum_{s=0}^{k'} \frac{4\lambda^{2r+1}\lambda^{2s+1}}{(2r+1)!\,(2s+1)!} [P^{2s+1}(P^{2r+1}(f, g), h) + P^{2s+1}(P^{2r+1}(g, h), f)$$
$$+ P^{2s+1}(P^{2r+1}(h, f), g)] = 0. \qquad (3.13)$$

Consider the highest-order term in the expansion ($s = r = k'$). In our comprehensive notation it reads

$$\frac{4\lambda^{4k'+2}}{\{(2k'+1)!\}^2} [h^{(2k'+1)}(f^{(4k'+2)} g^{(2k'+1)} + \ldots + f^{(2k'+1)} g^{(4k'+2)})$$
$$+ f^{(2k'+1)}(g^{(4k'+2)} h^{(2k'+1)} + \ldots + g^{(2k'+1)} h^{(4k'+2)})$$
$$+ g^{(2k'+1)}(h^{(4k'+2)} f^{(2k'+1)} + \ldots + h^{(2k'+1} f^{(4k'+2)})], \qquad (3.14)$$

each row containing $(2k+2)$ type of terms. Searching, as before, for types of terms containing $h^{(2k+1)}$ in rows 2 and 3 we find only one such type in each of these rows. So cancellation with the first row type of terms would occur only if $2k'+2 = 2$, i.e. $k' = 0$. For k even this means $k = 0$, for k odd it means $k = 1$. Once more we have discovered that our results of § 2 pertaining to the first-order deformation (2.1) are very special to that case. Any deformation to a finite order greater than one has no chance of being flexible or Lie admissible. Therefore it cannot fulfil Okubo's requirements for quantum mechanical calculations. On the contrary, to infinite order, Okubo's requirements are trivially met since the deformation product becomes associative.

We summarise our results in this section as follows. The deformation product defined by $(f, g)_{\lambda^k} = \Sigma_{r=0}^{k} (\lambda^r/r!)P^r(f, g)$, $k < \infty$, where P^r is the rth-order differential Poisson operator given by (2.5), has a non-vanishing associator which is of order λ^{k+1} or higher. Moreover, for $k > 1$ it is non-flexible and non-Lie admissible.

4. Conclusions

The basic result of our analysis is that Okubo's requirements are fulfilled for the deformation product (3.1) only if $k = 1$ (always restricting ourselves to k finite). We would like to discuss the implications of the departure from Okubo's requirements. In particular, we would like to know what is the meaning of finite-order calculations in the framework of the deformation product under consideration.

Clearly, any approximate calculation performed to a finite order leaves a remainder reflecting the neglected effects. The quality of this remainder may have profound consequences with respect to the total solution. For example, consider the case of two infinitesimal rotations performed around different axes, say $(1 + \varepsilon L_1)$ and $(1 + \varepsilon L_2)$. For convenience, we take the same (infinitesimal) angle for both rotations. It is usually stated in textbooks that the sequence by which two infinitesimal rotations are made does not matter. Explicitly

$$(1 + \varepsilon L_1)(1 + \varepsilon L_2) = 1 + \varepsilon(L_1 + L_2) + \mathrm{O}(\varepsilon^2),$$
$$(1 + \varepsilon L_2)(1 + \varepsilon L_1) = 1 + \varepsilon(L_1 + L_2) + \mathrm{O}(\varepsilon^2),$$

so that

$$(1+\varepsilon L_1)(1+\varepsilon L_2)-(1+\varepsilon L_2)(1+\varepsilon L_1)=0.$$

What has been neglected here is the term $\varepsilon^2(L_2L_1-L_1L_2)$, i.e. the commutator of generators L_1, L_2. This 'neglected' factor shows up when considering finite rotations; the latter do not commute. By a similar token, the neglected terms in our case (meaning the terms of order λ^{k+1} or higher) seem to be pathological not because they are non-associative but because they do not satisfy Okubo's requirements (let us call them non-Okubo terms for short). This means that finite order, $k>1$, calculations within the deformation scheme will leave as remainder non-Okubo terms, which do not have a precise quantum mechanical meaning. One cannot be certain about the cumulative effects of such terms. Our findings establish *a posteriori*, modulo an *a priori* demand for associativity, the wisdom involved in going to the exponential (star product). Within the framework of the star product one is confident that one is not leaving behind (i.e. neglecting) effects that are nonsensical to the very theory one is applying.

On the other hand, let us return to the previously raised question: what happens if a calculation cannot be carried to infinite order in $\hbar$, equivalently, when the quantum mechanical problem is not exactly solvable? It seems to us that it would be worthwhile to search for different realisations of deformation products, within the framework of the general definition (1.1), which might fulfil Okubo's requirement to finite order and which would coincide with the star product in the infinite-order limit.

It would be interesting to study restrictions imposed on the $C_k(f,g)$ by Okubo's requirements and relate such restrictions to the ones imposed on the $C_k(f,g)$ by Joseph's quantisation prescription. In this way, there may result a consistent realisation of the deformation (1.1) which could be suitable for finite-order calculations.

Another point of interest concerns the characterisation by Bayen *et al* (1978) of certain subsets of complex-valued phase-space functions as good observables. The latter seem to play a central, as well as special, role with respect to quantisation. These good observables could very well constitute the classical counterpart of Okubo's set of quantum mechanical observables $\mathscr{A}_0$.

We have already pointed out the significance of the subalgebra of good observables (Ktorides and Papaloucas 1982), with respect to Joseph's (1970) analysis concerning the quantisation mapping. Once more, one is led to speculate whether Okubo's framework provides a genuine extension which could accommodate Joseph's keen observations regarding the quantisation problem. Investigations along these lines are currently in progress.

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